UNIVALENCE CRITERIA IN MULTIPLY-CONNECTED DOMAINS

BY

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Dedicated to the memory of Zeev Nehari

ABSTRACT. Theorems due to Nehari and Ahlfors give sufficient conditions for the univalence of an analytic function in relation to the growth of its Schwarzian derivative. Nehari's theorem is for the unit disc and was generalized by Ahlfors to any simply-connected domain bounded by a quasiconformal circle. In both cases the growth is measured in terms of the hyperbolic metric of the domain. In this paper we prove a corresponding theorem for finitely-connected domains bounded by points and quasiconformal circles. Metrics other than the hyperbolic metric are also considered and similar results are obtained.

1. Introduction. In 1949 Z. Nehari [13] established a close connection between the univalence of a function f, analytic in the unit disc B, and the growth of its *Schwarzian derivative*,

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{1}$$

THEOREM 1. *In order that the function f, analytic in B, be univalent in B it is necessary that*

$$|S_f(z)| \leq 6/\left(1 - |z|^2\right)^2, \quad z \in B, \tag{2}$$

and sufficient that

$$|S_f(z)| \leq 2/\left(1 - |z|^2\right)^2, \quad z \in B. \tag{3}$$

The necessary condition seems to have first appeared in [9] and is shown to be sharp by taking f to be the Koebe function. E. Hille [8] observed that 2 is the best possible constant in (3). Much later L. Ahlfors and G. Weill [2] proved that if the number 2 is replaced by $2k$, $k < 1$, in (3) then the function has a quasiconformal extension to the entire plane. Then in 1963 Ahlfors [1] obtained a significant generalization of these results to a broad class of simply-connected domains. Before stating this theorem it is necessary to recall several definitions.

A Jordan curve in the extended complex plane $\overline{C} = C \cup \{\infty\}$ is said to be a K-quasiconformal circle if it is the image of the unit circle under a K-quasiconformal mapping of $\overline{C}$. The curve is said to be quasiconformal if it is K-quasiconformal

for some finite K and the number

$$K(C) = \inf\{K : f : \overline{C} \to \overline{C} \text{ is } K\text{-quasiconformal and } f(\partial B) = C\}$$

is often called the *dilatation* of C. Analytic Jordan curves are quasiconformal circles, but only circles and lines are 1-quasiconformal.

Next, if D is a simply-connected domain in $\overline{C}$ with at least three boundary points and g is any conformal mapping of D onto B then the function

$$\rho_D(z) = |g'(z)| / \left(1 - |g(z)|^2\right), \qquad z \in D,$$

is independent of the choice of conformal mapping and defines the density of the *hyperbolic metric* in D. If φ is analytic in D we define its *hyperbolic norm* to be

$$\|\varphi\|_D = \sup_{z \in D} |\varphi(z)|\rho_D(z)^{-2}. \tag{4}$$

Ahlfors' theorem can now be stated.

THEOREM 2. *If ∂D is a K-quasiconformal circle then there is a positive constant a, depending only on K, such that f is univalent in D and has a quasiconformal extension to $\overline{C}$ whenever f is analytic in D with $\|S_f\|_D \leqslant a$.*

Note that if $\|S_f\|_D$ is bounded then f is locally univalent in D and hence S_f as defined in (1) has an analytic extension to any possible poles of f and to ∞ if $\infty \in D$.

If D is an arbitrary domain in $\overline{C}$ of hyperbolic type then we can take the unit disc to be its universal covering surface. If $p: B \to D$ is any analytic projection then the density of the hyperbolic metric in D is defined by

$$\rho_D(p(z))|p'(z)| = \rho_B(z), \qquad z \in D.$$

Once again this is independent of the choice of p. The hyperbolic norm of an analytic function in D is defined just as in (4).

Our main result is as follows.

THEOREM 3. *Suppose that D is a finitely-connected domain and that each component of ∂D is either a point or a quasiconformal circle. Then there is a positive constant a, depending only on D, such that f is univalent in D whenever f is analytic in D with $\|S_f\|_D \leqslant a$.*

The proof of Theorem 3 proceeds by decomposing D into simply-connected subdomains in a way that makes it possible to apply Theorem 2. Such decompositions are discussed in §2 where a proof of Theorem 3 will be given. The proof depends on a lemma which will be proved in §3. In §4 we discuss some related results concerning metrics other than the hyperbolic metric and also consider the question of quasiconformal extension. §5 is devoted to constructing an example of an infinitely-connected domain for which the conclusion of Theorem 3 is valid.

2. Quasiconformally decomposable domains. In order to apply Theorem 2 to prove Theorem 3 we shall show that any two points in a domain D satisfying the hypotheses Theorem 3 can be placed in a simply-connected subdomain of D

bounded by a quasiconformal circle. To this end we introduce the following definition.

We say that a collection $\mathfrak{D}$ of domains $\Delta \subset D$ is a *K-quasiconformal decomposition* of D if each $\Delta \in \mathfrak{D}$ is bounded by a *K*-quasiconformal circle and if any two points $z_1, z_2 \in D$ lie in the closure of some $\Delta \in \mathfrak{D}$. We shall refer to this second condition as the *covering property* of $\mathfrak{D}$. We say that D is quasiconformally decomposable if it has a *K*-quasiconformal decomposition for some finite *K*. A domain may have many different quasiconformal decompositions with different constants *K* and the minimum such constant will depend on the domain.

Certainly any simply-connected domain bounded by a *K*-quasiconformal circle is *K*-quasiconformally decomposable. A more general converse also holds and will be proved in §4.

The following lemma is an immediate consequence of the definition.

LEMMA 1. *If D has a K_1-quasiconformal decomposition $\mathfrak{D}$ and if f is a K_2-quasiconformal mapping of $\overline{C}$ then $f(D)$ has a $K_1 K_2$-quasiconformal decomposition $\mathfrak{D}' = \{ f(\Delta) : \Delta \in \mathfrak{D} \}$.*

We now state the key lemma used for the proof of Theorem 3.

LEMMA 2. *If D is a finitely-connected domain and each component of ∂D is either a point or a quasiconformal circle, then D is quasiconformally decomposable.*

The proof of this lemma will be given in §3 where an explicit decomposition will be constructed. Lemma 1 will be used to reduce the problem to constructing a decomposition of a circle domain.

Assuming Lemma 2, the only other fact we require to prove Theorem 3 is a monotonicity property of the hyperbolic norm.

If $D' \subset D$ and φ is analytic in D then

$$\|\varphi\|_{D'} \leqslant \|\varphi\|_D. \tag{5}$$

This follows from the well-known property of the hyperbolic density; if $D' \subset D$ then $\rho_D \leqslant \rho_{D'}$.

PROOF OF THEOREM 3. By Lemma 2 D is quasiconformally decomposable and we suppose that $\mathfrak{D}$ is a *K*-quasiconformal decomposition where $K < \infty$ depends on D. Let z_1, z_2 be any two distinct points in D. Then there is a $\Delta \in \mathfrak{D}$ with $z_1, z_2 \in \overline{\Delta}$. Now by Theorem 2 there is a positive constant $a = a(K)$ such that $\|S_f\|_\Delta \leqslant a$ implies that $f|\Delta$ is univalent and has a quasiconformal extension to $\overline{C}$. In particular we see that f is actually univalent in $\overline{\Delta} \cap D$ and this covers the situation when either or both of z_1, z_2 lie on $\partial \Delta$.

Therefore if we choose a in this way and suppose that $\|S_f\|_D \leqslant a$ then from (5) it follows that $f(z_1) \neq f(z_2)$ and hence f is univalent in D.

It is possible to consider other metrics on D for which the associated norm is monotonic with respect to the domain and for which Theorem 2 holds. Thus the property of quasiconformal decomposability and the above argument will yield univalence criteria in these cases as well. Such alternate formulations of Theorem 3

will be considered in §4. Also in §4 we shall show how the constant a can be chosen so as to imply that f has a quasiconformal extension to $\overline{\mathbf{C}}$.

3. Proof of Lemma 2. The proof will be given in several stages, but the basic idea is this. By joining the boundary components of D in a specified manner we can write D as the union of simply-connected subdomains bounded by quasiconformal circles. The required decomposition will then be obtained by taking various unions of these subdomains. Ultimately we shall reduce the construction to the case where the joining is done by piecewise analytic arcs. We therefore begin this section with the following lemma.

LEMMA 3. *If C is a Jordan curve consisting of finitely many analytic arcs meeting at pairwise nonzero angles, then C is a quasiconformal circle.*

We omit the proof; see [12, pp. 97–105].

As a first step in the proof of Lemma 2 we show that isolated points are, in a sense, removable for quasiconformally decomposable domains. This will allow us to concentrate only on the case of nondegenerate boundary components.

LEMMA 4. *If D is a K-quasiconformally decomposable domain and if E is a set of n points in D, then $D \smallsetminus E$ is K_1-quasiconformally decomposable where K_n depends only on K and n.*

PROOF. We prove this by induction on n. First suppose that E consists of the single point ζ and set $D' = D \smallsetminus \{\zeta\}$. Let $\mathcal{D}$ be a K-quasiconformal decomposition of D. We construct a quasiconformal decomposition $\mathcal{D}'$ of D' from the sets in $\mathcal{D}$ in the following manner.

By definition there is at least one $\Delta \in \mathcal{D}$ with $\zeta \in \overline{\Delta}$. Define

$$\mathcal{D}_0 = \{\Delta \in \mathcal{D} : \zeta \in \partial\Delta\}.$$

If $\mathcal{D}_0 = \mathcal{D}$ then we can take $\mathcal{D}' = \mathcal{D} = \mathcal{D}_0$, and hence D' is also K-quasiconformally decomposable.

If this is not the case then there is a $\Delta \in \mathcal{D}$ with $\zeta \in \Delta$. Since $\partial\Delta$ is a K-quasiconformal circle there is a K-quasiconformal mapping g of $\overline{\mathbf{C}}$ such that $g(\partial B) = \partial\Delta$, and we may further assume that $g(B) = \Delta$ and $g(0) = \zeta$. We form three subdomains of $B' = B \smallsetminus \{0\}$;

$$B_1 = \{z \in B' : 0 < \arg(z) < 4\pi/3\}, \quad B_2 = e^{(2\pi i/3)}B_1, \quad B_3 = e^{(4\pi i/3)}B_1.$$

Thus each B_j is two thirds of the punctured disc.

It is easy to see that $\{B_1, B_2, B_3\}$ is a quasiconformal decomposition of B'. Indeed if $z_1, z_2 \in B'$ then they are at most diametrically opposite and so both z_1 and z_2 lie in the closure of at least one of the B_j. Furthermore by Lemma 3, ∂B_1 is a quasiconformal circle. We shall show in §5 that actually $K(\partial B_1) \leqslant 8/3$.

Now using Lemma 1 we see that the set $\{g(B_j) : j = 1, 2, 3\}$ is a K_1-quasiconformal decomposition, $\mathcal{D}_\Delta$, of $\Delta \smallsetminus \{\zeta\}$, where $K_1 = \frac{8}{3}K$. For each Δ in the set

$$\mathcal{D}_1 = \{\Delta \in \mathcal{D} : \zeta \in \Delta\}$$

we can construct the same type K_1-quasiconformal decomposition $\mathcal{D}_\Delta$. Thus if we now define

$$\mathcal{D}' = \{\Delta \in \mathcal{D} : \zeta \notin \Delta\} \cup \bigcup_{\Delta \in \mathcal{D}_1} \mathcal{D}_\Delta$$

then $\mathcal{D}'$ is a K_1-quasiconformal decomposition of D'.

From here the induction proceeds easily. Suppose that the theorem holds for E of cardinality $n-1$ and consider $D \setminus \{\zeta_1, \ldots, \zeta_n\}$. By the induction hypothesis there is a K_{n-1}-quasiconformal decomposition of $D \setminus \{\zeta_1, \ldots, \zeta_{n-1}\}$. Using the above construction we obtain an $\frac{8}{3}K_{n-1}$-quasiconformal decomposition of $(D \setminus \{\zeta_1, \ldots, \zeta_{n-1}\}) \setminus \{\zeta_n\}$. This completes the proof with $K_n = (\frac{8}{3})^n K$.

We now construct a quasiconformal decomposition of D when ∂D consists of a finite number of quasiconformal circles. By the preceding lemma this will be sufficient to include the case when ∂D also contains a finite number of point components. In order to reduce the problem to a simpler situation we employ a result due to G. Springer [14].

LEMMA 5. *Suppose that D and D' are two domains of the same finite connectivity and that the boundary components of each are quasiconformal circles. Then any quasiconformal mapping of D onto D' extends to a quasiconformal mapping of $\overline{\mathbf{C}}$.*

If we now map D conformally onto a circle domain D', i.e., a domain all of whose boundary components are circles in the finite plane, then by Lemmas 1 and 5 it is sufficient to construct a quasiconformal decomposition of D'. We therefore suppose that D itself is a finitely-connected circle domain.

First consider the case when D is a circular ring domain, which we may further assume to be the annulus

$$D = \{z : 1 < |z| < R\}.$$

The construction of the quasiconformal decomposition of D is virtually the same as that of $B \setminus \{0\}$ in the proof of Lemma 4. We define

$$D_1 = \{z \in D : 0 < \arg(z) < 4\pi/3\}, \quad D_2 = e^{(2\pi i/3)} D_1, \quad D_3 = e^{(4\pi i/3)} D_1.$$

From Lemma 3 we have that ∂D_1 is a quasiconformal circle and if $z_1, z_2 \in D$ then both points lie in the closure of at least one of the D_j just as before.

Now let D be a circle domain with boundary circles $C_0, \ldots, C_{n-1}$, $n \geqslant 3$. To construct a quasiconformal decomposition in this final case we map D conformally onto another canonical domain. Let $w = \Psi(z) = \Psi(z; u, v)$ be a conformal mapping of D onto the domain D' consisting of the entire plane minus n rectilinear slits lying on rays from the origin. The radial slit mapping Ψ has $\Psi(u) = 0$, $\Psi(v) = \infty$ and is determined up to a multiplicative constant. Regarding boundary behavior, Ψ has an analytic extension to $\overline{D}$ and the two points on C_j that correspond to the endpoints of $\Psi(C_j)$ divide C_j into two arcs each of which is mapped in a one-to-one fashion onto the slit. Thus we see that the preimage of a line segment meeting a slit $\Psi(C_j)$ will be an analytic arc forming a nonzero angle where it meets the circle C_j. This is certainly clear at the interior points of the slit and at the endpoints we shall

only be concerned with a segment lying on the ray containing $\Psi(C_j)$ where again the conclusion is immediate.

The idea is to join the slits cyclically and then join the endpoints farthest from the origin to ∞ along the rays. This will partition D' into simply-connected subdomains, the preimages of which will form part of the quasiconformal decomposition of D. To carry this out we require the following lemma.

LEMMA 6. *For each* $v \in D$ *it is possible to choose* $u \in D$ *so that no two slits lie on the same ray in the image of* D *under* $\Psi(z; u, v)$.

PROOF. We shall actually show slightly more, namely that the set of points $u \in D$ for which the assertion is false is nowhere dense in D. This will follow from several important identities in [5]. We use the same notation appearing there and indicate only what is needed in our situation.

First let $\Phi(z; u, v)$ be a conformal mapping of D onto the entire plane minus n concentric circular slits. Again $\Phi(u) = 0$, $\Phi(v) = \infty$ and the mapping is determined up to a multiplicative constant. The boundary behavior of Φ is analogous to that of Ψ. Next, for $z \in C_j, j = 0, \ldots, n-1$, define

$$\log \Phi(z; u, v) = \kappa_j(u, v) + ir_j(z; u, v),$$

$$\log \Psi(z; u, v) = s_j(z; u, v) + i\lambda_j(u, v),$$

$$k_j(u, v) = \kappa_j(u, v) + i\lambda_j(u, v).$$

Under suitable normalizations of Φ and Ψ it is shown in [5] that for each j, $k_j(u, v)$ is an analytic function of both u and v. Furthermore, for each $v \in D$ the function

$$R_{ij}(u) = \exp(k_i(u, v) - k_j(u, v)), \qquad i \neq j,$$

maps D onto an annulus minus $n - 2$ concentric circular slits with C_i and C_j corresponding to the two boundary circles.

From these results it is easy to obtain the desired conclusion. Fix $v_0 \in D$ and for $i, j = 0, \ldots, n-1, i \neq j$, define

$$E_{ij} = \{ u \in D : \lambda_i(u, v_0) = \lambda_j(u, v_0) \}.$$

We claim that E_{ij} is nowhere dense in D. If, to the contrary, $\overline{E}_{ij}$ contained a nonempty open set U then

$$k_i(u, v_0) - k_j(u, v_0) = \kappa_i(u, v_0) - \kappa_j(u, v_0)$$

would be real on U and therefore reduce to a constant. In view of the mapping properties of R_{ij} defined above this is clearly absurd. Finally set

$$E = \bigcup_{i \neq j} E_{ij}, \qquad i, j = 0, \ldots, n-1.$$

Then E is nowhere dense in D and the lemma is proved.

Now choose u and v as in the lemma and let D', C'_j denote the images of D and C_j under $\Psi(z; u, v)$. By renumbering, if necessary, we assume that $C'_0, \ldots, C'_{n-1}$ are in order of increasing argument. For each j let ξ_j be the endpoint of C'_j farthest from the origin and let Q'_j be the part of the ray containing C'_j that joins ξ_j to ∞. The Q'_j meet at well determined nonzero angles at ∞. The rays containing the C'_j

partition the plane into n sectors $S_0', \ldots, S_{n-1}'$ numbered so that S_j' lies between C_j' and C_{j+1}'. (We consider all subscripts to be reduced modulo n.) Denote the midpoint of C_j' by ω_j and let P_j' be a polygonal arc joining ω_j to ω_{j+1} that, except for its endpoints, lies completely in S_j'. Then

$$P' = \bigcup_{j=0}^{n-1} P_j'$$

is a closed polygon separating 0 from ∞ that does not intersect any of the Q_j'. Let G_0' and G_1' be the components of $D' \setminus P'$ that contain 0 and ∞ respectively. Observe that for each j, $S_j' \cap G_0'$ is a simply-connected Jordan domain. Now define

$$\Delta_{0j}' = G_0' \cup S_j', \quad \Delta_j' = D' \setminus \overline{\Delta}_{0j}',$$
$$\mathcal{D}' = \{\Delta_{0j}' : j = 0, \ldots, n-1\} \cup \{\Delta_j' : j = 0, \ldots, n-1\}.$$

Although the Δ_j', Δ_{0j}' are not Jordan domains, they are all simply-connected. We show that $\mathcal{D}'$ has the covering property for D' by distinguishing three possibilities for the position of two points $w_1, w_2 \in D'$. First, if $w_1, w_2 \in G_0'$ then w_1, w_2 lie in each Δ_{0j}'. Second, if $w_1 \in G_0'$ and $w_2 \in \overline{G}_1'$ then w_1, w_2 lie in the closure of some Δ_{0j}'. Finally suppose that $w_1, w_2 \in \overline{G}_1'$. If $w_1, w_2 \in \overline{S}_j'$ then $w_1, w_2 \in \overline{\Delta}_{0j}'$. Otherwise $w_1 \in \overline{S}_j'$ and $w_2 \in \overline{S}_i'$ with $j \neq i$. Since $n \geqslant 3$ there is a $k \neq i, j$ such that w_1 and w_2 lie in the closure of $D' \setminus \overline{\Delta}_{0k}' = \Delta_k'$.

We will denote the various parts of D corresponding under Ψ^{-1} to those of D' by the same symbol without the prime. Thus we define

$$\mathcal{D} = \{\Delta_{0j} : j = 0, \ldots, n-1\} \cup \{\Delta_j : j = 0, \ldots, n-1\}.$$

We claim that $\mathcal{D}$ is the required quasiconformal decomposition of D. In view of the above, $\mathcal{D}$ has the covering property. Under Ψ^{-1} the slits open up into full circles and G_0 is a Jordan domain bounded by the piecewise analytic Jordan curve consisting of the arcs P_j joined successively by the arcs of the C_j lying between the two points which map to the midpoints ω_j. (That is, joined by the arcs of the C_j that do not contain $\Psi^{-1}(\xi_j)$, the preimage of the farthest endpoint.) Likewise, S_j is bounded by the piecewise analytic curve consisting of the preimages of the rays containing C_j' and C_{j+1}'. Since $G_0 \cap S_j$ is a Jordan domain the same is true of $\Delta_{0j} = G_0 \cup S_j$. In fact Δ_{0j} is bounded by the piecewise analytic curve consisting of $Q_j \cup Q_{j+1}$, $\bigcup_{i \neq j} P_i$ and the arcs of the C_i that join these pieces. Furthermore, as noted earlier, at the C_i the arcs forming $\partial \Delta_{0j}$ meet at pairwise nonzero angles. Since all other angles are nonzero we conclude from Lemma 3 that $\partial \Delta_{0j}$ is a quasiconformal circle for each j. Similar remarks apply to the domains $\Delta_j = D \setminus \overline{\Delta}_{0j}$. Hence if we set

$$K = \max\{K(\partial \Delta) : \Delta \in \mathcal{D}\} < \infty$$

then $\mathcal{D}$ is a K-quasiconformal decomposition of D. This completes the proof of Lemma 2.

4. Other metrics and related results. If D is any proper subdomain of $\mathbf{C}$ and φ is analytic in D then we define its *quasihyperbolic norm* to be

$$\|\varphi\|_D^* = \sup_{z \in D} |\varphi(z)| \operatorname{dist}(z, \partial D)^2$$

where $\operatorname{dist}(z, \partial D)$ is the ordinary Euclidean distance. It follows from Schwarz's lemma that

$$\rho_D(z) \leqslant \operatorname{dist}(z, \partial D)^{-1}, \qquad z \in D,$$

and hence

$$\|\varphi\|_D^* \leqslant \|\varphi\|_D. \tag{6}$$

If D is simply-connected then the Koebe 1/4-Theorem gives the lower bound

$$(4 \operatorname{dist}(z, \partial D))^{-1} \leqslant \rho_D(z), \qquad z \in D, \tag{7}$$

and so for simply-connected domains we have the double inequality

$$\|\varphi\|_D^* \leqslant \|\varphi\|_D \leqslant 16\|\varphi\|_D^*. \tag{8}$$

Lower bounds for ρ_D in terms of the distance to ∂D in the case when D is multiply-connected have been recently obtained in [4].

Once again the quasihyperbolic norm is monotonic;

$$\|\varphi\|_{D'}^* \leqslant \|\varphi\|_D^* \tag{9}$$

whenever $D' \subset D$ and φ is analytic in D.

Because of (8) and (9), Theorem 3 continues to hold for the quasihyperbolic norm.

THEOREM 4. *Suppose that D is a finitely-connected proper subdomain of $\mathbf{C}$ and that each component of ∂D is either a point or a quasiconformal circle. Then there is a positive constant a, depending only on D, such that f is univalent in D whenever f is analytic in D with $\|S_f\|_D^* \leqslant a$.*

PROOF. Let $\mathcal{D}$ be a K-quasiconformal decomposition of D. From (8) and (9)

$$\|S_f\|_\Delta \leqslant 16\|S_f\|_\Delta^* \leqslant 16\|S_f\|_D^*$$

for any $\Delta \in \mathcal{D}$. Now choose $a = a(K) > 0$ in Theorem 2 such that $\|S_f\|_\Delta \leqslant 16a$ implies that $f|\Delta$ is univalent and has a quasiconformal extension to $\overline{\mathbf{C}}$. Then $\|S_f\|_D^* \leqslant a$ implies that f is univalent in D as before.

At this point we introduce domain constants associated with the two norms. (Compare [10].) If $D \subset \overline{\mathbf{C}}$ we define

$$\sigma_1(D) = \sup\{\|S_f\|_D \colon f \text{ is univalent in } D\}.$$

$$\sigma_2(D) = \sup\{a \colon \|S_f\|_D \leqslant a \text{ implies that } f \text{ is univalent in } D\}.$$

When $D \subset \mathbf{C}$ the constants $\sigma_1^*(D)$ and $\sigma_2^*(D)$ are defined the same way using the quasihyperbolic norm. From now on whenever the quasihyperbolic norm is involved we shall assume without further comment that ∞ does not lie in D.

As an example we have that $\sigma_1(B) = 6$, $\sigma_2(B) = 2$. O. Lehto [11] has shown that if D is simply-connected then $\sigma_1(D) \leqslant 12$ and has given an example for which

equality holds. Recently, A. F. Beardon and F. W. Gehring [3] have shown more generally that $\sigma_1(D) \leqslant 12$ for any domain D. Lehto's example in the simply-connected case shows that the constant 12 cannot be improved, and Beardon and Gehring construct examples to show that 12 is also the best possible constant for each class of domains of a fixed connectivity $n = 1, 2, 3, \dots$.

It was shown in [6] that for any domain D, $\sigma_1^*(D) \leqslant 6$ and in fact that $\sigma_2^*(B) = 6$. For the constant σ_2^* we can easily show that $\sigma_2^*(B) = \frac{1}{2}$. To see this first note that

$$\frac{1}{2} \operatorname{dist}(z, B)^{-2} = \frac{1}{2} \frac{1}{(1 - |z|)^2} \leqslant \frac{2}{(1 - |z|^2)^2} = 2\rho_B(z)^2, \qquad z \in B,$$

and, as $\sigma_2(B) = 2$, we have $\sigma_2^*(B) \leqslant \frac{1}{2}$. Next, the function $f(z) = (1 - z)^{i\alpha}$, α small and positive, is not univalent in B and has

$$S_f(z) = \frac{1 + \alpha^2}{2} \frac{1}{(1 - z)^2}, \qquad z \in B,$$

which shows that $\sigma_2^*(B) \geqslant \frac{1}{2}$ as well.

Finally we remark that from (6) it follows that $\sigma_1^*(D) \leqslant \sigma_1(D)$ and $\sigma_2^*(D) \leqslant \sigma_2(D)$. Hence Theorem 4 is slightly stronger than Theorem 3.

The constants σ_2 and σ_2^* are related to an important topological property of a domain [6].

A set E in $\overline{\mathbf{C}}$ is said to be *b-locally connected*, $b \in (1, \infty)$, if for all z_0 and $r > 0$, points in $E \cap \overline{B}(z_0, r)$ can be joined in $E \cap \overline{B}(z_0, br)$ and points in $E \setminus B(z_0, r)$ can be joined in $E \setminus B(z_0, r/b)$.

Here $B(z_0, r)$ denotes the disc with center z_0 and radius r. Two points $z_1, z_2 \in E$ are joined in E if there is an arc $C \subset E$ having z_1 and z_2 as endpoints.

The importance of this concept for plane domains lies in the following lemma, proved in [6].

LEMMA 7. *If D is b-locally connected then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on b.*

The connections with the domain constants are these.

LEMMA 8 [6]. *If $\sigma_2^*(D) > 0$ then D is b-locally connected with*
$$b \geqslant \max(5/\sigma_2^*(D) + 1, 3).$$

LEMMA 9 [3]. *If $\sigma_2(D) > 0$ then D is b-locally connected with*
$$b \geqslant \max(20\pi/\sigma_2(D) + 1, 3).$$

These results allow us to deduce the following characterization.

THEOREM 5. *For finitely-connected domains D the following five properties are equivalent.*
   (i) $\sigma_2(D) > 0$.
   (ii) *D is b-locally connected.*
   (iii) *∂D consists of points and quasiconformal circles.*
   (iv) *D is quasiconformally decomposable.*
   (v) $\sigma_2^*(D) > 0$.

PROOF. Lemmas 9, 7, and 2 combine to give (i) implies (ii) implies (iii) implies (iv). That (iv) implies (v) is the proof of Theorem 4 and (v) implies (i) because $\sigma_2^*(D) \leqslant \sigma_2(D)$ as noted earlier.

REMARK. Lemma 9 is proved in [3] not in terms of the hyperbolic metric but in terms of the reduced Bergmann kernel function $K_D(z, \bar{z})$ of D. However, as is seen in the course of the proofs, one can just as well consider the hyperbolic metric to obtain the same conclusions. On the other hand, the sufficient conditions for univalence stated here in Theorems 2, 3, and 4 could be recast using the kernel function. Ahlfors' theorem (Theorem 2) remains true with the kernel function simply because

$$K_D(z, \bar{z}) = \rho_D(z)^2/\pi, \qquad z \in D,$$

when D is simply-connected. The kernel function has the same monotonicity property with respect to the domain as the hyperbolic density and thus the same proof for Theorem 3 can be given using the kernel function with changes in terminology only.

We close this section by considering the connection between the constants σ_2, σ_2^* and quasiconformal extension of conformal mappings. To do this it is necessary to use a transformation property of the two norms. If f and g are analytic functions then whenever their composition makes sense we have the formula

$$S_{f \circ g} = (S_f \circ g)(g')^2 + S_g.$$

In particular if f is analytic in D and g is a conformal mapping of D then we can write

$$S_f - S_g = (S_{(f \circ g^{-1})} \circ g)(g')^2$$

and from the conformal invariance of the hyperbolic metric it follows that

$$\|S_f - S_g\|_D = \|S_{f \circ g^{-1}}\|_{g(D)}. \tag{10}$$

To determine a corresponding formula for the quasihyperbolic norm we require a simple bound for the derivative of a conformal mapping of an arbitrary domain $D \subset \mathbf{C}$. If g is univalent in D then

$$|g'(z)| \leqslant 4 \frac{\text{dist}(g(z), \partial g(D))}{\text{dist}(z, \partial D)}, \qquad z \in D.$$

This follows from the Koebe $1/4$-Theorem by considering for each $z_0 \in D$ and for $\zeta \in B$ the function

$$F(\zeta) = \frac{g(r\zeta + z_0) - g(z_0)}{rg'(z_0)}$$

where $r = \text{dist}(z_0, \partial D)$. Using this we find

$$\|S_f - S_g\|_D^* \leqslant 16 \|S_{f \circ g^{-1}}\|_{g(D)}^*. \tag{11}$$

We can now prove an extension theorem associated with σ_2 and σ_2^*.

THEOREM 6. *Suppose that D is a finitely-connected domain with $\sigma_2(D) > 0$. If f is analytic in D with $\|S_f\|_D < \sigma_2(D)$ or $\|S_f\|_D^* < \sigma_2^*(D)$ then f has a quasiconformal extension to $\overline{\mathbf{C}}$.*

PROOF. By Theorem 5 each component of ∂D is either a point or a quasiconformal circle. From the definition, if $\|S_f\|_D < \sigma_2(D)$ or $\|S_f\|_D^* < \sigma_2^*(D)$ then f is univalent in D so that any point components of ∂D are removable singularities. By Lemma 5 to show that f has a quasiconformal extension to $\overline{\mathbf{C}}$ it suffices to show that the nondegenerate components of $\partial f(D)$ are quasiconformal circles. To do this we shall apply Theorem 5 to $f(D)$.

First assume that $\|S_f\|_D < \sigma_2(D)$ and suppose that g is analytic in $f(D)$ with

$$\|S_g\|_{f(D)} < \sigma_2(D) - \|S_f\|_D.$$

Now set $h = g \circ f$. Then from (10), $\|S_h - S_f\|_D = \|S_g\|_{f(D)}$ or

$$\|S_h\|_D \leqslant \|S_f\|_D + \|S_g\|_{f(D)} < \sigma_2(D).$$

Therefore h is univalent in D and consequently g is univalent in $f(D)$. But then

$$\sigma_2(f(D)) \geqslant \sigma_2(D) - \|S_f\|_D > 0$$

and the desired conclusion follows from Theorem 5.

If instead we assume that $\|S_f\|_D^* < \sigma_2^*(D)$ then we suppose that g is analytic in $f(D)$ with

$$\|S_g\|_{f(D)}^* < \tfrac{1}{16}\big(\sigma_2^*(D) - \|S_f\|_D^*\big).$$

Using the same argument it follows from (11) that

$$\sigma_2^*(f(D)) \geqslant \tfrac{1}{16}\big(\sigma_2^*(D) - \|S_f\|_D^*\big) > 0.$$

Lehto [10] proved a special case of Theorem 6 for simply-connected domains using results from the theory of Teichmüller spaces.

5. An example. In this section we give an example of an infinitely-connected domain for which both σ_2 and σ_2^* are positive. As in the previous discussion this will be shown by constructing an explicit quasiconformal decomposition. In this case there will be an infinite number of elements in the decomposition and their boundary curves will consist of an infinite number of arcs, but by employing an auxiliary quasiconformal mapping we can reduce the problem of estimating the dilatations to an appeal to Lemma 3. We begin this section by constructing this mapping and as a corollary we shall obtain the estimate in Lemma 4.

Let $C = C(\alpha)$ be the circle that intersects the unit circle ∂B at the points ± 1 at an angle α, $0 < \alpha < \pi/2$, and extends above the unit circle. We refer to C as a protecting α-circle. Let E be the union of B and the disc bounded by C and let B^- denote the lower half of B.

LEMMA 10. *There is a $(1 + \pi/2\alpha)$-quasiconformal mapping F_α of $\overline{\mathbf{C}}$ such that $F_\alpha(B) = B^-$ and F_α is the identity in $\overline{\mathbf{C}} \smallsetminus E$.*

PROOF. Map B onto the upper half-plane by $T(z) = i(1 + z)/(1 - z)$. Then C is mapped to a line passing through the first and third quadrants at an angle α while

B^- corresponds to the first quadrant. We define a folding of the plane by

$$f_\alpha(re^{i\theta}) = f_\alpha(r, \theta) = \begin{cases} \left(r, \dfrac{\theta}{2}\right), & 0 \leqslant \theta \leqslant \pi, \\[2mm] \left(r, \left(1 + \dfrac{\pi}{2\alpha}\right)\theta - \dfrac{\pi}{2\alpha}(\pi + \alpha)\right), & \pi \leqslant \theta \leqslant \pi + \alpha, \\[2mm] (r, \theta), & \pi + \alpha \leqslant \theta \leqslant 2\pi. \end{cases}$$

We also define $f_\alpha(0) = 0$, $f_\alpha(\infty) = \infty$. Then f_α is a homeomorphism of $\overline{\mathbf{C}}$ and we easily compute the dilatation of each piece of f_α to be 2, $1 + \pi/2\alpha$, and 1 respectively. Since $0 < \alpha < \pi/2$ we have $1 + \pi/2\alpha > 2$. Since the lines $\theta = 0$ and $\theta = \alpha$ are of σ-finite linear measure we conclude by a theorem of K. Strebel [15] that f_α is a $(1 + \pi/2\alpha)$-quasiconformal mapping of $\overline{\mathbf{C}}$. The required mapping F_α is now defined by $F_\alpha = T^{-1} \circ f_\alpha \circ T$.

For use in our construction we note that any disc can be folded by conjugating F_α with a Möbius transformation.

COROLLARY 1. *For any half-disc D we have $K(\partial D) \leqslant 2$.*

PROOF. Without loss of generality we can take $D = B^-$. The estimate now follows from

$$K(\partial B^-) \leqslant \inf\{1 + \pi/2\alpha : \alpha \in (0, \pi/2)\} = 2.$$

COROLLARY 2. *If $B_1 = \{z \in B : 0 < \arg(z) < 4\pi/3\}$ then $K(\partial B_1) \leqslant \frac{8}{3}$.*

PROOF. The mapping $g(r, \theta) = (r, 3\theta/4)$ is a $\frac{4}{3}$-quasiconformal mapping of $\overline{\mathbf{C}}$ that maps B_1 onto the top half of B. The conclusion is then a consequence of Corollary 1.

For applications of a similar type of folding in three dimensions see [7].

We now construct the example. Let $\{B_n\}$ be a sequence of discs with $\overline{B}_n \subset B$, with disjoint closures and with hyperbolic centers $\{x_n\}$, $n = 0, 1, 2, \ldots$ on the real axis. For definiteness we assume that $0 = x_0 < x_1 < \cdots$. We further assume that the discs are chosen to have the same finite hyperbolic area and so that the hyperbolic distance between any two adjacent discs is a constant $\lambda > 0$. For symmetry we set $x_{-n} = -x_n$ and let B_{-n} be the reflection of B_n in the imaginary axis. We define the domain D to be

$$D = B \smallsetminus \bigcup_{n=-\infty}^{\infty} \overline{B}_n.$$

A quasiconformal decomposition of D is constructed as follows. The segments formed by the intersection of D with the real axis join the B_n successively and, by assumption, each has hyperbolic length λ. We let β_n be the segment joining B_n to B_{n+1}, $n = \ldots, -1, 0, 1, \ldots$. Next, for each n let L_n be the hyperbolic line through x_n that is symmetric with respect to the real axis. Let γ_n be the one of the two arcs formed by $D \cap L_n$ that lies in the lower half-plane. Let D^+ and D^- be the parts of D lying in the upper and lower half-planes respectively. Then the γ_n partition D^- into a sequence of disjoint subdomains D_n^-, bounded laterally by γ_n and γ_{n+1}. The domains D^+ and D_n^- meet at the segment β_n and so $\Delta_n = D^+ \cup \beta_n \cup D_n^-$ is

again simply-connected. We now define

$$\mathcal{D} = \{D^-\} \cup \{\Delta_n : n = \ldots, -1, 0, 1, \ldots\}.$$

$\mathcal{D}$ has the covering property, for if $z_1, z_2 \in D$ then there are only three possibilities to consider for their position. Either they both lie in $\overline{D}^+$, both lie in $\overline{D}^-$ or one lies in D^+ and one in D^-. In any of these cases both points lie in the closure of an element in $\mathcal{D}$.

The Δ_n are all congruent since $\Delta_n = T_n(\Delta_0)$ where T_n is the Möbius transformation of the unit disc onto itself that fixes ± 1 and maps 0 to x_n. Therefore the dilatations of their boundary curves are all the same and we need only show that $\partial\Delta_0$ is a quasiconformal circle. As it stands now Lemma 3 is not applicable since ∂D^- and $\partial\Delta_0$ consist of an infinite number of circular arcs and line segments, although they do meet at right angles by construction. However, the folding mapping of Lemma 10 will enable us to flatten the semicircles arising from the B_n while leaving the rest of the boundary unchanged. This is so precisely because the folding was constructed to be the identity outside the disc and its protecting circle, a fact that will also allow us to fold all of the discs simultaneously.

It is necessary to provide each B_n with a protecting α-circle that passes through the vertices of B_n on the real axis, extends above B_n and lies inside B. Furthermore we wish to choose a single α_0 so that adjacent α_0-circles are at most tangent. Since the α_0-circle on B_n can be obtained from the one on B_0 by applying T_n, this will be the case if this α_0-circle extends no farther to the right (and left) than the hyperbolic midpoint of β_0 (β_{-1}) and no farther above B_0 than the point i on the imaginary axis. A simple calculation shows that $\alpha_0 > 0$ must be chosen so as to satisfy respectively

$$\sec\alpha_0 \leqslant \frac{\tanh(r + \lambda/2)}{\tanh r}, \qquad \sec\alpha_0 + \tan\alpha_0 \leqslant (\tanh r)^{-1},$$

where r is the hyperbolic radius of B_0. We combine these and choose $\alpha_0 > 0$ so that

$$\sec\alpha_0 + \tan\alpha_0 \leqslant \frac{\tanh(r + \lambda/2)}{\tanh r}.$$

Let E_n be the union of B_n and the disc bounded by its protecting α_0-circle and let F_n be the associated folding of B_n onto its lower half. We define a mapping F of $\overline{\mathbf{C}}$ onto itself by

$$F(z) = \begin{cases} z, & z \in \overline{\mathbf{C}} \setminus \bigcup_{n=-\infty}^{\infty} E_n, \\ F_n(z), & z \in E_n. \end{cases}$$

Because F_n is the identity on $\overline{\mathbf{C}} \setminus E_n$ we see that F is a homeomorphism. Moreover, for each n, $F|_{E_n}$ is a $(1 + \pi/2\alpha_0)$-quasiconformal mapping and because $\bigcup_{n=-\infty}^{\infty} \partial E_n$ is of σ-finite linear measure we conclude as before ([15]) that F is a $(1 + \pi/2\alpha_0)$-quasiconformal mapping of the entire plane.

The image of D^+ under F is just the upper half of the unit disc and since $K(\partial D^+) = K(\partial D^-)$ it follows from Corollary 1 that $K(\partial D^-) \leqslant 2(1 + \pi/2\alpha_0)$. The image of Δ_0 under F is the mushroom shaped domain bounded by the upper

semicircle, the real axis from -1 to 1 minus the segment β_0, the arcs of ∂B_0, ∂B_1 from the endpoints of β_0 to γ_0, γ_1 and the part of the lower half of ∂B joining γ_0 and γ_1. Since all arcs meet at right angles we can now apply Lemma 3 to deduce that $\partial\Delta_0$ is a quasiconformal circle.

We remark that by using several more foldings it is possible to give an estimate for $K(\partial\Delta_0)$ but because the calculations are somewhat more involved we do not give them here. We also note that by using the same techniques it is possible to give a slightly more general example. Namely, we require that the discs B_n, still centered on the real axis, are chosen so as to satisfy the following geometric condition. Let R_n be the ring domain bounded by ∂B and ∂B_n and for $m \neq n$ let R_{mn} be the ring domain in $\overline{\mathbf{C}}$ bounded by ∂B_m and ∂B_n. We assume that there is a positive constant b such that

$$\inf_n \operatorname{mod} R_n \geqslant b, \qquad \inf_{m,n} \operatorname{mod} R_{mn} \geqslant b$$

where "mod" denotes the conformal modulus.

These conditions have two consequences both of which were exemplified in the preceding construction. First, there is a finite constant $\eta = \eta(b)$ such that $\sup_n A(B_n) \leqslant \eta$ where $A(B_n)$ denotes the hyperbolic area of B_n in B. Second, there is a positive constant $\lambda = \lambda(b)$ such that $\rho(B_n, B_{n+1}) \geqslant \lambda$ where $\rho(B_n, B_{n+1})$ denotes the hyperbolic distance between B_n and B_{n+1}. We shall not pursue this here, but we do point out that in [3] an example of an infinitely-connected domain is given for which the various univalence criteria fail by choosing the B_n so that $\sup_n A(B_n) = \infty$.

Note. The author has recently learned of a related paper, *Injectivity theorems in plane and space* by O. Martio and J. Sarvas to appear in Ann. Acad. Sci. Fenn. Ser. AI. The authors consider injectivity theorems related to the Schwarzian derivative but from a different point of view. They say that a domain D is a uniform domain if there is a positive constant a such that each pair of points $x, y \in D$ can be joined by a rectifiable curve γ in D with

$$\min(s(x, z), s(z, y)) \leqslant a \operatorname{dist}(z, \partial D), \qquad z \in \gamma,$$

and

$$s(x, y) \leqslant a|x - y|.$$

Here $s(x, z)$ denotes the distance along γ from x to z. The authors then show that $\sigma_2^*(D) > 0$ for a uniform domain. This does not contain Theorem 4 of this paper since it would be necessary to establish that a domain is uniform if it is bounded by a finite number of points and quasiconformal circles. Actually, Martio and Sarvas prove a theorem in the opposite direction.

In fact it is true that a plane domain is uniform if and only if it is quasiconformally decomposable. For this and related results we refer to a forthcoming paper, *Uniform domains and the quasi-hyperbolic metric* by the present author and F. W. Gehring to appear in J. Analyse Math.

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